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TWO THEOREMS ON CONICS.

By S. Lefschetz.

1. This paper contains two unrelated theorems on conics, of which the first is virtually contained in a proposition given by Poncelet,* while the second is the equivalent for conics of a well-known theorem on rational quartics, and so far as we know is new under this form. To both propositions we apply considerations of number of projective conditions satisfied by a given configuration.

2. The first theorem is the following: *When two complete quadrilaterals circumscribed to a conic have a common diagonal, their vertices, not situated on the latter, are on another conic.*

Let OAB be the triangle of reference, and

$$S = x^2 - yz = 0,$$

the equation of a conic S tangent to OA and OB in A and B. The two tangents drawn from the point a $(0, 1, \lambda)$ are simultaneously represented by:

$$4\lambda(x^2 - yz) + (\lambda y + z)^2 = 0,$$

which can be written:

$$4\lambda x^2 - (\lambda y - z)^2 = 0.$$

By changing λ into μ, we obtain the tangents drawn from the point b $(0, 1, \mu)$, so that

$$4\lambda x^2 - (\lambda y - z)^2 + \kappa[4\mu x^2 - (\mu y - z)^2] = 0 \qquad (1)$$

represents a conic going through the intersections of the two pairs of tangents. Similarly a conic going through the points of intersection of the tangent from the point a' $(0, 1, \lambda')$ with the tangents from the point b' $(0, 1, \mu')$ will be represented by

$$4\lambda' x^2 - (\lambda' y - z)^2 + \kappa'[4\mu' x^2 - (\mu' y - z)^2] = 0 \qquad (2)$$

and we must show that for proper values of κ and κ', the equations (1) and (2) will represent the same curve.

Such values will satisfy the system:

$$\frac{\lambda + \kappa\mu}{\lambda' + \kappa'\mu'} = \frac{\lambda^2 + \kappa\mu^2}{\lambda'^2 + \kappa'\mu'^2} = \frac{1 + \kappa}{1 + \kappa'},$$

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* Traité des propriétés projectives des figures, T. 1, § 568. This was pointed out to the writer by E. Fabry.

which effectively can be solved for κ and κ'. Thus we get

$$\kappa = -\frac{(\lambda - \lambda')(\lambda - \mu')}{(\mu - \lambda')(\mu - \mu')}$$

and substituting in (1) we obtain:

$$2(\lambda\mu - \lambda'\mu')(2x^2 - yz) + [\lambda\mu(\lambda' + \mu') - \lambda'\mu'(\lambda + \mu)]y^2 + [(\lambda + \mu) - (\lambda' + \mu')]z^2 = 0$$

as the equation of the conic Σ circumscribed to the two quadrilaterals circumscribed to S. Now the pairs of points (a, b) and (a', b') determine on AB an involution, the parameters of which are defined by:

$$\alpha\lambda\mu + \beta(\lambda + \mu) + \gamma = 0,$$
$$\alpha\lambda'\mu' + \beta(\lambda' + \mu') + \gamma = 0,$$

which gives for the equation of Σ:

$$\Sigma = 2\beta(2x^2 - yz) - \gamma y^2 - \alpha z^2 = 0,$$

which shows that Σ depends solely on the involution on AB.

3. Let C and D be the double points of the involution just considered on AB, they clearly determine entirely the position of Σ with respect to S, and if E is one of the points of intersection of OC and the conic S, then the whole system is projectively determined by the four points O, A, B, E, and the cross ratio $(ABCD)$. Hence projectively there are ∞^1 systems such as that formed by S and Σ. Since there are projectively ∞^2 systems of two conics, there must be one relation between the invariants of the system (S, Σ). Effectively, if D is the discriminant of

$$S + \rho\Sigma = 0$$

we have:

$$D = \rho^3\Delta_1 + \theta_1\rho^2 + \theta_2\rho + \Delta_2$$
$$= -[\tfrac{1}{4}\rho^3 + 2\beta\rho^2 + (5\beta^2 - \alpha\gamma)\rho + 4\beta(\beta^2 - \alpha\gamma)],$$

Δ_1, Δ_2 being the discriminants of S and Σ, and θ_1, θ_2 their relative invariants. Identifying we get:

$$\Delta_1 = -\tfrac{1}{4}, \quad \theta_1 = -2\beta, \quad \theta_2 = -(5\beta^2 - \alpha\gamma),$$
$$\Delta_2 = -4\beta(\beta^2 - \alpha\gamma).$$

As these four form a complete invariant system of two conics, and only three of them contain β and $(\alpha\gamma)$, by eliminating these two parameters we can obtain one and only one relation. We have:

$$4\theta_2\Delta_1 = 5\beta^2 - \alpha\gamma = \tfrac{5}{4}\theta_1^2 - \alpha\gamma$$

$$\frac{8\Delta_2\Delta_1^2}{\theta_1} = \beta^2 - \alpha\gamma = \theta_1^2 - \alpha\gamma.$$

Hence the relation desired is*

$$\theta_1^3 - 4\Delta_1\theta_1\theta_2 + 8\Delta_2\Delta_1^2 = 0,$$

expressing the condition that there exist a quadrilateral circumscribed to S and inscribed to Γ. If we put $(ABCD) = \sigma$, then $\sigma = m_1/m_2$, m_1 and m_2 being the roots of

$$-\alpha m^2 + 2\beta m + \gamma = 0.$$

Hence σ is given by the equation

$$\frac{\alpha\gamma}{4\beta^2}(\sigma + 1)^2 - \sigma = 0$$

and if

$$J = \frac{\Delta_1\Delta_2}{\theta_1\theta_2},$$

is the absolute invariant of the system, then

$$\sigma^2 + 2\frac{(14J - 3)}{10J - 1}\sigma + 1 = 0.$$

4. The second proposition can be stated thus: *If the tangents at three of the common points of a conic and a triangle go through the opposite vertices, then the triangle is self polar with respect to the conic.*†

There are two possible cases according as the three points are, or are not on different sides.

In the first case let ABC be a triangle, a_1, a_2, b_1, b_2, c_1, c_2 its intersections with a conic S, the tangents at a_1, b_1, c_1 going respectively through the vertices A, B, C. Let D be the intersection of Bb_1 and Cc_1, and E and F, the points where b_1c_1 meets BC and AD respectively. We have $(EFc_1b_1) = -1$ and hence AD is polar of E with respect to S. Hence the polar of A goes through E, and as it goes also through a_1, it must be Ea_1, or BC. Similarly for B and C, which proves the proposition in this case.

In the second case suppose that S is tangent to Bb_1, Bb_2, Cc_1. Then B

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* The expression given is an invariant since it is homogeneous and of the 9th degree in the coefficients of two general conics.

† Transforming quadratically with respect to the triangle we obtain this: *If a rational quartic has more than three flecnodes, it has necessarily three biflecnodes.* For proofs see: Basset, Treatise on cubic and quartic curves, p. 111; J. E. Rowe, "A complete system of invariants of the rational quartic," Transactions of the American Mathematical Society, Vol. 12, p. 309.

is pole of AC and therefore the polar of c going through B and c_1 must be Bc_1 or AB. Since C and B are poles of the opposite sides the proposition is true in this case too.

5. Another proof will be given here. The system formed by a triangle and an arbitrary conic depends upon three absolute invariants, for it is a degenerate quintic with nine double points, and since the most general curve of the fifth order depends upon 12 absolute invariants, between which each new double point establishes a relation, the truth of our statement follows. Otherwise also: It is easily shown that the system is projectively completely defined by the three cross ratios (ABc_1c_2), (BCa_1a_2), (CAb_1b_2). That there is no relation between them is seen thus: To any four points a_1, a_2, b_1, b_2 taken arbitrarily on the proper sides, there corresponds a pencil of conics which determines on AB an involution, a pair of which will be defined in one of two ways by (ABc_1c_2), which is thus shown to be arbitrary when the other two cross-ratios are given.

Now if we impose upon the system the conditions that one or two of the tangents to the conic from the vertices meet it on the sides, they alone will do so, and the conditions are independent. For there is a conic tangent to Aa_1 in a_1, going through B and tangent to an arbitrary line at its intersection with AB, and for this conic Aa_1 is the only tangent from the vertices meeting it on the sides. Next, among the conics tangent to Aa_1 in a_1, and to Bb_1 and b_1, there is one tangent to AB at a certain point K, with respect to which b_1K and a_1K will be the polars of A and B, and therefore again Aa_1, and Bb_1 are the only tangents from the vertices which will meet this conic on the sides. The same is evident for conics tangent to Aa_1 and Aa_2 in a_1 and a_2, since for them AB and AC are arbitrary.

If now we impose upon a conic the conditions that three of the tangents at the intersections with the sides of the triangle go through the opposite vertices, these conditions being projective and independent, will determine the cross ratios on the sides. Hence any two systems satisfying these conditions for which the cross ratios are definite, will be projectively equivalent. But such a system is presented by a conic and any one of its self-polar triangles, into which the system considered can therefore be projected —and this proves the proposition.

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